VETOR

Analysis of beams on elastic base via variational methods☆

Análise de vigas sobre base elástica via métodos variacionais

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Abstract

The study of beams is one of the main problems investigated in Civil Engineering, and these structures are governed by differential equations. This article seeks to identify numerical solutions of the balance equation of beams on elastic basis, using the Finite Element Method and applying the variational methods, i.e., Placement, Sub-regions and Least Squares Method, aiming to compare the results obtained through numerical experiments and the analytical solution, to identify the variational method that provides the best approximate solution, befitting the analytical solution. This is a bibliographic review, with descriptive approach and numerical simulations using the Phyton programming language. We compared the solutions of the model problem for two different cases, using the methods mentioned above, noting that in the 1st case, the Methods of Sub-regions and Placement provide the best approximation for vertical displacements, with a polynomial base function, while in the 2nd case the trigonometric function provides a better approximation, presenting significant variations in relation to the 1st case, due to changes in parameters, spring coefficient (k), modulus of longitudinal elasticity (E) and cross-sectional inertia (I). Thus, starting from this formulation, other problems frequently encountered in engineering can be analyzed, such as continuous beams and dynamic analysis of beams.

Keywords

Numerical Analysis • Beams • Variational Methods

Resumo

O estudo de vigas é um dos principais problemas investigados na Engenharia Civil, sendo estas estruturas regidas por equações diferenciais. Este artigo busca identificar soluções numéricas da equação de equilíbrio de vigas sobre base elástica, utilizando o Método dos Elementos Finitos e aplicando os métodos variacionais, a saber, Colocação, Sub-regiões e Método dos Mínimos Quadrados, visando comparar os resultados obtidos através de experimentações numéricas e a solução analítica, para identificar o método variacional que fornece a melhor solução aproximada, condizente com a solução analítica. Trata-se de uma revisão bibliográfica, com abordagem descritiva e realização de simulações numéricas utilizando a linguagem de programação Phyton. Comparamos as soluções do problema modelo para dois casos diferentes, utilizando os métodos citados anteriormente, constatando que no 1° caso, os Métodos das Sub-regiões e Colocação fornecem a melhor aproximação para os

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deslocamentos verticais, com uma função base polinomial, enquanto no 2° caso a função trigonométrica fornece uma melhor aproximação, apresentando variações significativas em relação ao 1° caso, devido as mudanças nos parâmetros, coeficiente de mola (K), módulo de elasticidade longitudinal (E) e inércia da seção transversal (I). Assim, partindo desta formulação, podem ser analisados outros problemas encontrados frequentemente na engenharia, tais como vigas contínuas e análise dinâmica de vigas.
Palavras-chave

Análise numérica • Vigas • Métodos Variacionais

1 Introduction

In general, studies involving structures allow the use of several classical techniques to work with problems in the field of constructions. The study of beams is considered as one of the main problems investigated in Civil Engineering, where depending on the type of structure addressed, the problems have occurred quite frequently. These structures are governed by differential equations with various behaviors and properties. In fact, there are methodologies for the development of mathematical models, where it is possible to perceive the existence of a set of parameters that define the dynamics of the adopted model.

As addressed by Santos and Lacerda [1], vertical displacement monitoring and internal efforts are initial parameters used for the sizing and control of structures in general. However, there are several types of structures, and in this work the approach is limited to the beams of elastic bases, where the predominance of the elastic base in its entire tying, causes certain behaviors that directly influence the structural projects of the buildings.

In this sense, to analyze the behavior of elastic base beams, it is necessary to approach some concepts, such as the calculation of vertical displacement, which requires the application of advanced mathematical methods, among which there is the Finite Element Method (FEM), which allows, in a variational way, the discretization of the solution domain in small regions and approximation of the behavior of variables (unknowns) in these regions. Thus, the application of the MEF aims to generate approximate values for the observed behavior using approximation of derivatives by means of finite elements, to the desired order of error [2].

So, it is important to emphasize that the Variational Methods allow to obtain approximate solutions to the solution of a certain problem of contour value, assuming that the functions considered are sufficiently regular, in the sense that the integration or derivation operations have felt, limited exclusively to the case of linear operators. The Finite Element method provides a general and systematic technique for the construction of form functions, which are designated to assume the unit or zero value at the nodal points within each element [3].

Polycarpou [4] states that the main idea behind the method is the representation of the domain in smaller subdomains called finite elements, and the form functions must be a complete set of polynomials, whose accuracy of the solution depends, among other factors, on the order of these polynomials, which can be linear, quadratic, cubic or higher order.

Thus, this work seeks to identify the numerical solution of the balance equation of beams on elastic basis, using the Finite Element Method for different domain approaches, with application of some variational methods, namely that of placement, sub-regions and least squares method, aiming to compare the results obtained with the analytical solution, in order to identify which method provides the best approximate solution, befitting the analytical solution.

2 Mathematical Modeling

The beams are defined by item 14.4.1.1 of NBR 6118 [5], defines the beams as being "linear elements in which flexion is preponderant". There are different types and models of beams, but this work focuses on the descriptive approach of beams on elastic base. According to Santos and Lacerda [1], in a beam on elastic foundation that is influenced by external loads, the reaction forces of the foundation are proportional at each point to the displacement of the elastic line. Thus, in this type of model the distributed loads that act on the beam can be admitted as being a uniform distribution along its length.

2.1 Problem definition

The model problem under study is a beam supported along its entire length by an elastic medium and subjected to a vertical loading that acts on the plane of symmetry of the cross section, where the distributed loads acting on the foundation can be admitted as evenly distributed, according to Fig. 1, besides being understood as $R(x)$ being the support reaction of the beam on an elastic base, where the support base can be the ground, which provides the displacement reaction of the beam $u(x)$.

Therefore, the reactions in the elastic medium can be expressed by Eq. (1), evidencing a constant of proportionality k, known as elastic constant, which is defined as the ratio between the support reaction $R(x)$ and the displacement reaction $u(x)$ along the length of the beam. In this way, you get:

$$R(x) = k.u(x)\,. \tag{1}$$

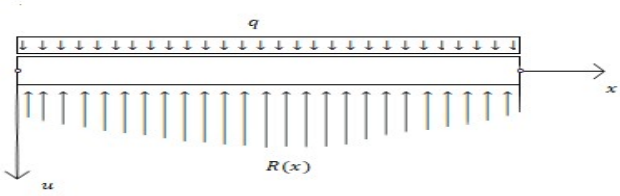


Figure 1: Uniformly Loaded Beam on Elastic Base.

Thus, it is understood that for each infinitesimal element dx there are elastic reaction effects $d[R(x)]$, which cause the displacement $d[u(x)]$ at each point of the beam. From where, the differential element dx is considered in equilibrium if the sum of vertical and deflector momentum efforts is null, and the following relationships are deduced when applying Eq. (1):

$$\begin{aligned} \frac{dQ}{dx} &= R(x) - q(x), \\ \frac{dQ}{dx} &= ku(x) - q(x). \end{aligned} \tag{2}$$

On the other hand, the sum of moments in the vertical is null, which is divided by dx results in:

$$\begin{aligned} &\frac{dM}{dx} - (Q + dQ) - q(x).\frac{dx}{2} + r(x).\frac{dx}{2} = 0, \\ &\frac{dM}{dx} - (Q + dQ) + [r(x) - q(x)]\frac{dx}{2} = 0. \end{aligned} \tag{3}$$

Now, replacing Eq. (2) in Eq. (3), the following result is reached:

$$\begin{aligned} &\frac{dM}{dx} - Q - \ dQ + \frac{dQ}{dx}.\frac{dx}{2} = 0, \\ &\frac{dM}{dx} = Q + \frac{dQ}{2} = q(x) \rightarrow \frac{dM}{dx} = q(x). \end{aligned} \tag{4}$$

In fact, there is:

$$\begin{aligned} &\frac{d^2u(x)}{dx^2} = -\frac{M}{EI} \rightarrow \frac{d}{dx}\left[\frac{d^2u(x)}{dx^2}\right] = \frac{d}{dx}\left(-\frac{M}{EI}\right) \rightarrow \frac{d^3u(x)}{dx^3} = -\frac{d}{dx}\frac{M}{EI}, \\ &\frac{d}{dx}\left[\frac{d^3u(x)}{dx^3}\right] = \frac{d}{dx}\left(-\frac{d}{dx}\frac{M}{EI}\right) \rightarrow \frac{d^4u(x)}{dx^4} = -\frac{1}{EI}\frac{d^2M}{dx^2} \rightarrow -EI\frac{d^4u(x)}{dx^4} = \frac{d^2M}{dx^2}. \end{aligned} \tag{5}$$

In addition,

$$\frac{dQ}{dx} = \frac{d^2M}{dx^2} = -EI\frac{d[u^4(x)]}{dx^4}. \tag{6}$$

And since the sum of vertical forces is null,

$$-EI\frac{d^4u(x)}{dx^4} = k.u - q(x) \rightarrow EI\frac{d^4u(x)}{dx^4} + k.u = q(x). \tag{7}$$

Moreover, from the mathematical point of view, to solve the fourth-order differential equation defined by Eq. (7) it is necessary to consider the boundary conditions that govern it, being necessary to know the nature of the ends and the load applied on the beam, which in the case under study is a beam supported with uniformly distributed load.

2.2 Analytical solution

According to Santos and Lacerda [1], Eq. (7) is commonly referred to as one-dimensional equation of the beam on elastic base, where $u = u(x)$ it is vertical displacement, $q = q(x)$ is the external loading, k is the spring constant associated with the elastic base, E is the modulus of longitudinal elasticity, I is the inertia of the cross section of the beam and x is the spatial coordinate, the length of the beam.

Assuming that the beam material is homogeneous, the inertia of the section is constant, the loading is evenly distributed, and the elastic base is constant and stable, it is possible to rewrite Eq. (7) as

$$\frac{d^4u(x)}{dx^4} + 4\left(\frac{k}{4EI}\right).u = \frac{q(x)}{EI}. \tag{8}$$

As can be seen, Eq. (8) is described in terms of constants and can be solved as a Linear Ordinary Differential Equation of Higher Order. By simplifying notation, the factor $\frac{k}{4EI}$ will be replaced by λ. Therefore:

$$\frac{d[u^4(x)]}{dx^4} + 4\lambda^4u(x) = \frac{q(x)}{EI} \rightarrow EI\frac{d[u^4(x)]}{dx^4} + 4\lambda^4u(x) = q(x)\ , \tag{9}$$

where

$$\lambda = \sqrt[4]{\frac{k}{EI}}. \tag{10}$$

Also, Eq. (9) can be expressed as follows:

$$u(x) = u_c(x) + u_p(x), \tag{11}$$

where u_c is the characteristic or homogeneous solution, and u_p the main or particular solution. The solution of equation (11) corresponds to the solution of the homogeneous equation,

$$\frac{d[u^4(x)]}{dx^4} + 4\lambda^4u(x) = 0. \tag{12}$$

In addition, Eq. (12) can be solved through a characteristic equation,

$$r^4 + 4\lambda^4u(x) = 0. \tag{13}$$

Thus, it is possible to determine some results by applying the integration techniques for Eq. (8) and if reaching the following results:

$$r_1 = \lambda(1+i), r_2 = \lambda(1-i), r_3 = \lambda(-1+i), r_4 = \lambda(-1-i) \tag{14}$$

Therefore, the roots found in Eq. (14) correspond to the characteristic solution u_c. Thus,

$$u_c(x) = e^{\lambda x}[A\,cos(\lambda x) + B\,sin(\lambda x)] + e^{-\lambda x}[C\,cos(\lambda x) + D\,sin(\lambda x)]. \tag{15}$$

In turn, the particular solution of the problem is obtained through a portion of the Eq. (9), which makes it a non-homogeneous differential equation. Considering the values q, E and I constants, Eq. (8) classifies the solution of the equation as a constant of null parameters, that is,

$$u(x) = x^s(A_m x^m + \ldots + A_1 x + A_0)e^{rs}. \tag{16}$$

As a condition of $m = r = s = 0$, it is necessary to:

$$u_p(x) = A_0. \tag{17}$$

Rewriting the Eq. (10), one has:

$$\lambda^4 = \frac{k}{4EI}. \tag{18}$$

Rewriting the Eq. (9) in terms of the $u_p(x)$:

$$u_p(x) + 4\lambda^4 \cdot u_p(x) = \frac{q(x)}{EI}. \tag{19}$$

Considering $u_p(x) = 0$ and replacing Eqs. (17) and (18) in Eq. (19), the following are obtained:

$$4A_0\lambda^4 = \frac{q(x)}{EI} \rightarrow A_0 = \frac{q(x)}{4\lambda^4 EI} = \frac{q(x)}{4\left(\frac{k}{4EI}\right)EI} = \frac{q(x)}{k} \rightarrow u_p(x) = A_0 = \frac{q}{k}, \tag{20}$$

with $q = q(x)$.

Replacing Eqs. (15) and (20) in Eq. (11), one finally obtains the analytical solution of the proposed problem, which is given by:

$$u(x) = \frac{q}{k} + e^{\lambda x}[A\,cos(\lambda x) + B\,sin(\lambda x)] + e^{-\lambda x}[C\,cos(\lambda x) + D\,sin(\lambda x)]. \tag{21}$$

The constants *A*, *B*, *C* and *D* are determined from the types of support of the beam analyzed, which delimit the respective boundary conditions of the analyzed problem. It is worth noting that the calculations of these constants come from the resolution of a system formed by 4 equations with 4 variables.

The boundary conditions described for this problem are given by:

$$u(0) = 0, u(L) = 0, \frac{d^2u(0)}{dx^2} = 0, \frac{d^2u(L)}{dx^2} = 0 \tag{22}$$

To impose these boundary conditions, it is necessary to calculate the second derivative of Eq. (21). Calculating the second derivative of the Eq. (21), we have:

$$\frac{d^2[u(x)]}{dx} = \frac{2\lambda^2[(-D + Be^{2\lambda x})\,cos(\lambda x) + (C - Ae^{2\lambda x})\,sin(e^{2\lambda x})]}{e^{\lambda x}} \tag{23}$$

By resolving the system of four nonlinear equations, you can obtain the following definition for these constants:

$$A = -\left[\frac{q(1 + e^{\lambda L}\,cos(\lambda L))}{k(1 + e^{2\lambda L} + 2e^{\lambda L}\,cos(\lambda L)}\right] \tag{24}$$

$$B = -\left[\frac{e^{\lambda L}q\,sin(\lambda L))}{k + e^{2\lambda L}k + 2e^{\lambda L}k\,cos(\lambda L)}\right] \tag{25}$$

$$C = -\left[\frac{e^{\lambda L}q(e^{\lambda L} + e^{\lambda L}\,cos(\lambda L))}{k(1 + e^{2\lambda L} + 2e^{\lambda L}\,cos(\lambda L)}\right] \tag{26}$$

$$D = -\left[\frac{e^{\lambda L}q\,sin(\lambda L))}{k + e^{2\lambda L}k + 2e^{\lambda L}k\,cos(\lambda L)}\right] \tag{27}$$

Replacing the results represented by Eqs. (24), (25), (26) and (27) in Eq. (21), the transverse displacements $u(x)$ obtained on a beam supported by elastic foundation, given by:

$$u(x) = \frac{q[cos(\lambda L) + cosh(\lambda L) - cos(\lambda L)\, cosh(\lambda(L-x)) - cos(\lambda(L-x))\, cosh(\lambda L)]}{k(cos(\lambda L) + cosh(\lambda L))} \qquad (28)$$

Therefore, Eq. (28) is the analytical solution of Eq. (7) for a beam supported on an elastic foundation.

2.3 Variational Methods

Variational Methods allow you obtaining approximate solutions for a given boundary value problem. It is assumed that the interpolation functions considered are sufficiently regular, in the sense that the integration or derivation operations have meaning, limited exclusively to the case of linear operators.

2.3.1 Weighted residue method

Let U and V normed and complete spaces, that is, in each space is associated with a standard and every sequence $u_{n_{n=1,\infty}}$ of elements $u_n \in U$, such that $||u_n - u_m|| \to 0$, $n \to \infty$ (Cauchy sequence) always converges to an element u_0 of the same space.

The following transformation is now defined:

$$S: U \to V$$
$$(u, v) \to S(u, v) = \int \Omega uvd\omega, \qquad (29)$$

which means that given an ordered pair (u, v), where $u \in U$ and $v \in V$, the transformation results in a real number.

Given a linear operator B and an element $f \in U$, we want to find the solution to the following linear problem $Bu = f$, where B can be the operator $B = \left(EA\frac{d}{dx}\right)$ or $B = \nabla$. We say that it $u \in U$ is the solution to the problem if it is found that

$$S(Bu - f, v) = 0, \forall v \in V \Rightarrow \int \Omega(Bu - f)vd\Omega = 0, \forall v \in V. \qquad (30)$$

According to Miranda [6], to obtain the approximate solution u_h of u, the weighted residuals method proposes the following algorithm:

1. Build a complete sequence $\Phi_{k_{k=1,n}}$ of functions that are sufficiently regular and meet all boundary conditions;
2. For every finite n, the set $\Phi_{k_{k=1,n}}$ must be linearly independent, that is, this set of functions is the basis of space $U_h = span\{\Phi_1, \Phi_2, \dots, \Phi_n\}$;
3. Take as u approximation the linear combination $u_h = \sum_{j=1}^{n} \alpha_j \Phi_j$, where the coefficients α_j, $j = 1, \dots, n$ will be determined later;
4. Build a sequence $\{w_{k_{k=1,n}}\}$, such that $V_h = span\{w_1, w_2, \dots, w_n\}$;
5. Calculate for finite n, the coefficients α_j so that the residue $r_h = Bu_h - f = \sum_{j=1}^{n} \alpha_j B\Phi_j - f$ satisfies

$$S(r_h, w_i) = \int \Omega(\textstyle\sum_{j=1}^{n} \alpha_j B\phi_J - f)w_i d\Omega = 0, \forall w_i \in V.$$

2.3.2 Placement Method

This method is a particular case of the Weighted Residue Method, in which functions w_i are functions $\delta - Dirac$ (δ_i) associated with points x_i, with $i = 1, 2, \dots, n$ from Ω. This function has the following property:

$$\int_\Omega h(x)\delta(x - x_i)\, d\Omega = h(x_i). \qquad (31)$$

With this, to $i = 1,2, \dots, n, \delta_i = \delta(x - x_i)$ and $u_h = \sum_{j=1}^{n} \alpha_j \Phi_j$, the method becomes ,

$$\int_\Omega r_h \delta_i \, d\Omega = (Bu_h - f)|_{x_i} = 0. \tag{32}$$

2.3.3 Sub-region method

It is a particular case of the Weighted Residue Method, which seeks to zero the residual function in all sub-regions of the domain. Be ${\Omega_i}^e$ a partition Ω of such that $\cup {\Omega_i}^e = \Omega$ and ${\Omega_i}^e \bigcap {\Omega_j}^e = \varnothing, \forall i, j$. We define w_i such that

$$w_i(x) = \begin{cases} 1 \text{ if } x \in {\Omega_i}^e \\ 0 \text{ if } x \notin {\Omega_i}^e. \end{cases} \tag{33}$$

With this, for $i = \{1, 2, \dots, N\}$, the method becomes:

$$\int_\Omega r_h w_i \, d\Omega = \int_{\Omega_i^e} r_h w_i \, d\Omega_i^e = \int (Bu_h - f) \, d\Omega_i^e = 0. \tag{34}$$

2.3.4 Least Squares Method

The Least Squares Method resolves a constraint imposed by another Variational Method, the Ritz Method, which required differential operator B to be symmetrical and positive defined. To do this, you define the following internal product:

$$\langle u, v \rangle_{B,B} = \int_\Omega BuBv \, d\Omega \Rightarrow \|u\|_{B,B}^2 = \langle u, v \rangle_{B,B}, \tag{35}$$

where $B = \left(EA \frac{d}{dx}\right)$ is the differential operator and $u \in U, v \in V$.

Once this standard is defined, we can put the problem of finding the best approximation to the solution u_0 of the problem $Bu_0 = f$ in Ω relation to the standard $||\,.\,||_{B,B}$ as being a problem of minimization of the functional, that is, u minimizes the functional

$$J(u) = ||u - u_0||^2{}_{B,B} \Rightarrow J(u_0) = \min J(u). \tag{36}$$

Considering

$$\begin{aligned} J(u) = \|u - u_0\|^2 = \langle u - u_0, u - u_0 \rangle_{B,B} = \langle u, u \rangle_{B,B} - 2\langle u, u \rangle_{B,B} + \langle u_0, u_0 \rangle_{B,B} \\ = \int_\Omega BuBu \, d\Omega - 2 \int_\Omega BuBu_0 \, d\Omega + \int_\Omega Bu_0 Bu_0 \, d\Omega, \end{aligned} \tag{37}$$

and, as u_0 it is the solution of the problem, we have:

$$\begin{aligned} J(u) = BuBud\Omega - 2 \int_\Omega Buf \, d\Omega + \int_\Omega ff \, d\Omega \\ = \int_\Omega (Bu - f)(Bu - f) \, d\Omega = \int_\Omega (Bu - f)^2 \, d\Omega. \end{aligned} \tag{38}$$

And, as the residual function is defined by $r = Bu - f$, the problem can be defined as minimizing the following functional:

$$J(u) = \int_\Omega r^2 \, d\Omega. \tag{39}$$

So to get the approximate solution, considering the coordinated functions in such a way that $u_h = \sum_{j=1}^{n} \alpha_j \Phi_j$, where Φ_j it is regular enough for $B\Phi_j$ it to make sense. With this, we have to:

$$J(u_h) = J(\alpha_j) = \int_\Omega r^2 \, d\Omega = \int_\Omega (B\Phi_j\alpha_j - f)^2 \, d\Omega. \tag{40}$$

Therefore, the problem is to minimize the functional $J(\alpha_j)$, that is, calculate the minimum of a functional α_j variable, thus falling into the resolution of the following system of equations, to $j = \{1,2,..,n\}$:

$$\int_{\alpha_j} r \frac{\partial r}{\partial \alpha_j} d\Omega. \tag{41}$$

Therefore, the discrete function u_{0_h} that minimizes the functional $J(u)$ is given by $u_{0_h} = \sum_{j=1}^{n} \alpha_j \Phi_j$

2.4 Interpolation functions

The following are the basic solution functions that can be used to obtain an approximate numerical solution.

2.4.1 Polynomial function

The polynomial function used to obtain an approximation based on the methods adopted is of the type:

$$\phi_n(x) = x^n(x-1), \forall n \in \{1,2,\dots,n\}, \tag{42}$$

$$\Phi(x) = x^{n+2}(x-L)^{n+2}. \tag{43}$$

The set of functions $\{\Phi_i\}_{i=1,2,\dots,N}$ is linearly independent and regular in the problem domain Ω. In addition, it satisfies the contour conditions defined by the problem:

$$\phi_n(0) = 0^n(0-1) = 0, \forall n \in \{1,2,\dots,n\}, \tag{44}$$

$$\Phi(L) = x^{n+2}(L-L)^{n+2} = 0, \forall n \in \{1,2,\dots,n\}. \tag{45}$$

2.4.2 Trigonometric function

The trigonometric function used to obtain an approximation based on the methods adopted is of the type:

$$\phi_n(x) = sin\left(\frac{\pi}{2}x\right)^n, \forall n \in \{1,2,\dots,n\}. \tag{46}$$

The set of functions $\{\Phi_i\}_{i=1,2,\dots,N}$ is linearly independent and regular in the problem domain Ω. In addition, it satisfies the contour conditions defined by the problem:

$$\phi_n(0) = sin\left(\frac{\pi}{2}0\right)^n, \forall n \in \{1,2,\dots,n\}. \tag{47}$$

3 Results and Discussions

The analysis of beams supported on elastic base, subject to the action of a uniformly distributed load was modeled through the application of finite element methods. The application of this technique took place through the implementation of the numerical methods already explained using the Python programming language, where the computational objects developed are associated with the processes produced at each stage of the formulation.

Thus, through numerical simulations obtained by the Variational Methods of Placement (MC), Sub-regions (MS) and Least Squares (MMQ), we performed a comparison between the results obtained for the coefficients α_j. An important factor to be emphasized is that for the application of these numerical methods for the solution of Eq. (7) it is necessary to consider the length of the beam (L), the longitudinal modulus of elasticity (E), the inertia of the cross section (I), the external loading applied on the beam (q) and the spring coefficient (k).

Thus, based on research, it is possible to affirm that the EI ratio represents the bending stiffness of the beam while the k represents the stiffness of the foundation soil. Therefore, for the application of these numerical resolution methods described above, only two cases were considered among the existing possibilities:

- **1st** Case: $L = 1.0, I = 1.0, E = 1.0, q = F = 1000.0, k = 54.0$

In this case, the application of MEF in the analysis of the beam subject to the action of a distributed load (q) was calculated based on the application of 11 points distributed throughout its structure. Thus, the values obtained for the coefficients for each method are indicated in Table 1, and it is important to highlight that in the Least Squares Method a different base function was used from that used in the other two methods.

Table 1: Comparison of the results obtained for the coefficient α_j.

Placement Method	Sub-region method	Least Squares Method
-0.124186043370565	-0.123437326372301	-0.61718657946408
0.496744173511415	0.493749298277676	-1.33312309627824
-0.837105924475265	-0.839373799829299	-1.62937278005176
0.772713164958089	0.789998944021680	-12.7601386868480
-12.0156101275190	-11.9701412936300	-35.1697972361481
23.3228998518319	23.1996583392921	-41.6666575044702
-18.3437668148348	-18.4669991651836	-25.3835459473802
6.72831170299304	6.91654620489500	-85.0718847036956
-78.5301384960302	-78.1553488355315	-78.1553435094945

It was possible to identify that the Placement Methods and Sub-regions present a better approximation for the coefficients and when compared with the Least Squares Method variations were observed. Regarding the numerical and exact solutions of the problem, comparative graphs of the solutions for each Variational Method were plotted, and it is important to highlight that the interpolation function used in each method follows the characteristics already explained in item 2.4.1, being sufficiently regular and satisfying all boundary conditions.

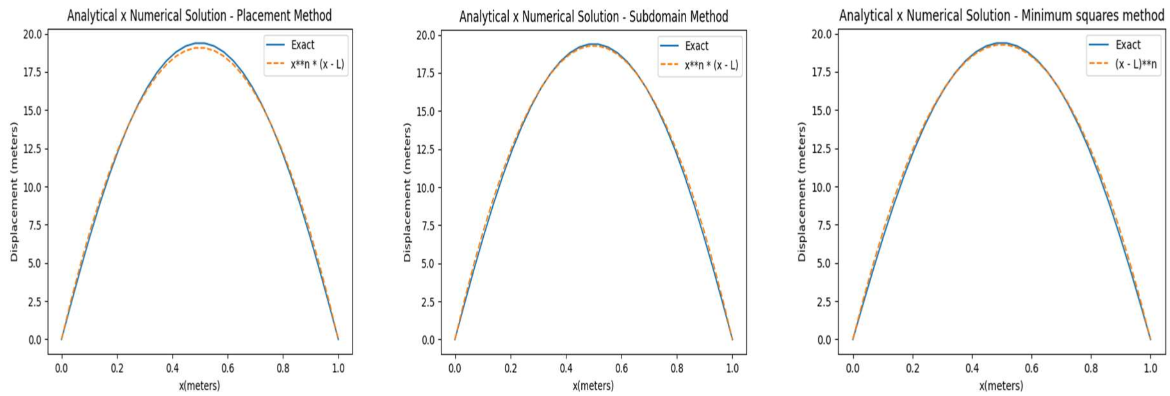


Figure 2: Comparison between analytical and numerical solution by placement method, sub-regions and least squares.

It is important to highlight that the interpolation function used in each method is Eq. (42), except for the Least Squares Method that was used equation $\Phi(x) = (x - L)^n$.

In Fig. 2 we have the numerical solutions for vertical displacement using the Placement Methods, Sub-regions and Least Squares Method compared with the analytical solution defined by Eq. (28) and its respective derivatives, and both variational methods present a good approximation when compared to the analytical solution.

- **2nd Case**: $L = 1.0, I^*E = 4.0 \times 10^6, q = F = 1000.0, k = 0.5$

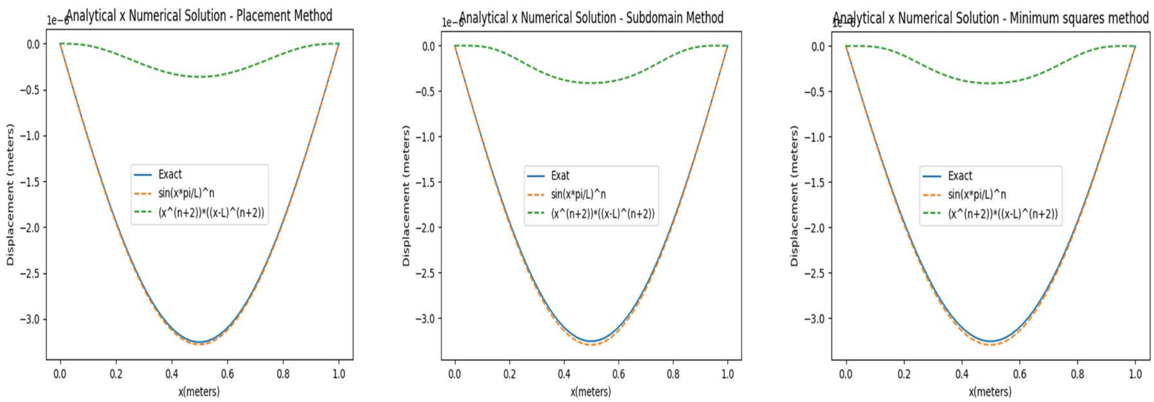
In this case, numerical simulations showed significant variations compared to the 1st case. Such variations have to do with the changes in the parameters K, E and I, which can be seen in the graphs shown in Fig. 3.

The values obtained for the coefficients in each method used are indicated in Table 2 below. The numerical solution was obtained based on two interpolation functions defined in Eqs. (43) and (46), whose characteristics have already been explained in items 2.4.1 and 2.4.2. In addition, they are regular and meet the boundary conditions imposed on the problem.

Table 2: Comparison of the results obtained for the coefficient α_j.

Placement Method	Sub-region method	Least Squares Method
2.07557912998504e-9	4.68081952392329e-5	-1.11575743116935e-8
-4.15115825997007e-9	0.000282219376581574	1.38882176769011e-8
0.0	-0.000452054074702820	4.17821534561348e-8
0.0	0.000906617102950374	-1.11575743116935e-7
-9.22808110718594e-8	0.000447871191183831	0.0
3.37579831276531e-6	2.083333333313307e-5	3.36198734119546e-6

When analyzing the data obtained in Table 2, it was not possible to identify which of the methods present a better approximation to the coefficients, due to the inconstancy of the results obtained.



Figures 3: Numerical Solution by Placement Method, Sub-regions and Least Squares.

Observing Fig. 3, it is perceived that the interpolation function that best approximates the analytical solution is the trigonometric function defined in Eq. (46). We found that the Variational Methods present a good approximate solution to the problem under study, when compared to the analytical solution. The Methods of Sub-regions and Least Squares provide the best approximation for displacement, and it is important to highlight that the variation of parameters: spring coefficient (k), applied load (q), longitudinal elasticity modulus (E), cross-sectional inertia (I) and spatial coordinate x, directly influence the solution, resulting in a better or worse result for the type of structure researched.

In addition, there is the question of the proposal of the discretized space model, to which the condition of loading application was imposed on points equally spaced along the beam structure supported on the elastic base. All these points are relevant to cross-sectional deformations, and are therefore indispensable for the dimensioning of such structure.

Thus, it is concluded that the results obtained with the Variational Methods and MEF for displacement values in two-set beams and elastic base were satisfactory, and that the method can be used in analyses of other problems frequently encountered in engineering.

4 Conclusions

In this work, a brief description of the main methodologies used in the resolution of beams in elastic foundation was carried out, in addition to a case study, where the results obtained for two application examples are exposed.

These types of problems worked occur with a certain frequency in the area of Civil Engineering, it is important to highlight that the quantities involved in the characterization of the behavior of these structural elements - displacements, deformations and efforts make up the descriptive mathematical model of such structures.

However, for the cases addressed, in the differential equation that governs the behavior of this type of structure, an additional term emerges that is associated with the vertical reaction exerted by the foundation on the beam and considering the difficulty associated with obtaining an analytical solution for the differential equation, “numerical tools” methods were described and applied that allow the achievement of approximate solutions.

These techniques are conceptually much simpler and although they only allow us to get an approximation, this approach can be as good as you want and from a certain limit is confused in practical terms with the analytical solution of the problem itself. With this, the analysis developed for the case of the beam supported on elastic base subject to the action of a uniformly distributed load occurred with the application of the Finite Element Method.

From the analysis of Figures 2 and 3, it is perceived that good results were obtained from the perspective of the variational methods and Finite Element Method, from the calculation of coefficients to the displacements in beams two-set on an elastic basis.

Thus, starting from this formulation, other problems frequently encountered in engineering can be analyzed, such as continuous beams and dynamic analysis of beams.

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